Generalized Constraints on Quantum Amplification.

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We derive quantum constraints on the minimal amount of noise added in linear amplification involving input or output signals whose component operators do not necessarily have c-number commutators, as is the case for fermion currents. This is a generalization of constraints derived for the amplification of bosonic fields whose components posses c-number commutators.

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It has often been noted that Heisenberg uncertainty relations, or the underlying operator commutation relations, impose performance restrictions on amplifiers[1]-[8] and detectors[9][10]. In particular it has been shown[6][7][11] that a linear phase insensitive amplifier must emit noise out of its output port with a noise power of at least $(G^2-1)\hbar\omega/2$ per unit bandwidth, where G^2 is the power gain. In laser amplifiers this noise results from spontaneous emission. In parametric amplifiers this noise is frequency converted and amplified Nyquist noise from the black body absorber terminating the idler port. The discussions have generally been carried out in the context of amplifying bosonic fields with no particle conservation law (simply called "bosonic" below), such as electromagnetic fields in optical systems or collective charge excitations in electronic circuits amplified by lasers or parametric amplifiers . Heisenberg operators representing such signals have the property that they can be decomposed into two components, the commutator of which is a c-number. Therefore, the Heisenberg principle is used to derive quantum constraints on their amplification in a way that is similar to arguments normally used for an ordinary pair of canonical variables. However, such an analysis excludes a very common (perhaps even the most common) case of a signal: one in which the commutator of the Heisenberg operators representing the components of the input signal is an operator instead of a c-number.

For example, in semiconductor or molecular transistors often the signal is carried by current of fermions. Due to the conservation law of these particles the current operators in this case are bilinear in the fermion creation and annihilation operators. As a result the commutators of the current operators are themselves operators bilinear in fermion creation and annihilation operators. Consequently, the derivations of the quantum limits of amplifier noise performance that have been provided for amplifiers of bosonic fields are not directly applicable to amplifiers amplifying fermion currents. To overcome this limitation we derive generalized quantum limits of amplifier noise performance that are applicable when the input or output fields are either fermion or boson fields.

We begin with a review of the usual analysis for bosonic amplifiers [1]-[7]. Suppose first that the signals can be described by a single pair of canonical variables: Let x and p be the position and momentum Heisenberg operator for the signal fed into the amplifier and let X and P denote the position and momentum operators for the amplified signal. Ideally, for a phase insensitive amplifier, one would like a device that simply produces an amplified copy of the input:

$$X = Gx \quad ; \quad P = Gp \ . \tag{1}$$

However, these relationships are incompatible with the position-momentum commutation relations that must be satisfied on each side of the amplifier:

$$[x,p] = i\hbar \quad ; \quad [X,P] = i\hbar \ . \tag{2}$$

Hence, Eqs. (1) and (2) must be modified according to

$$X = Gx + X_N \quad ; \quad P = Gp + P_N \ . \tag{3}$$

where X_N and P_N are noise operators. Since the noise is uncorrelated with the signal, one has

$$[x, X_N] = [p, X_N] = [x, P_N] = [p, P_N] = 0 \ . \tag{4}$$

Equations (2) through (4) impose the following commutation relation on the noise operators:

$$[P_N, X_N] = i\hbar(G^2-1) \ . \tag{5}$$

This immediately implies the uncertainty relation

$$\Delta X_N \Delta P_N \geq \frac{1}{2}\hbar(G^2-1) \tag{6}$$

where $\Delta A \equiv (\langle A^2\rangle - \langle A\rangle^2)^{1/2}$.

A similar argument to the above shows that a phase sensitive amplifier performing the transformation

$$X = Gx \quad ; \quad P = \frac{1}{G}p \tag{7}$$

is fully compatible with the commutation relations Eq. (2). Such an amplifier need not add noise to the signal delivered at the output port [6][7][12][13].

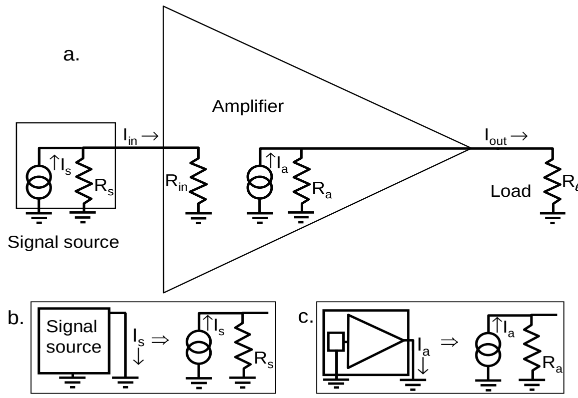

Figure 1: **a.** A source R_s feeding an amplifier connected to a load R_ℓ. **b.** The signal source Norton equivalent. The signal source would produces a current I_s if short-circuited. **c.** The amplifier Norton equivalent. The signal source + amplifier would produce a current I_a if short-circuited.

Consider now a signal carried by a current $I(t)$. Let us expand the operator $I(t)$ in Fourier form:

$$I(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega I(\omega) e^{-i\omega t} \tag{8}$$

For simplicity we shall consider systems small enough so that the spatial dependence of the current may be ignored. $I(t)$ is Hermitian and therefore: $I(\omega) = I^\dagger(-\omega)$.

To carry out an analysis analogous to that of Eq. (1) through (7) operators playing the role of x and p need to be constructed. To this end we introduce $I_f(t)$ which consists of $I(t)$ passed through a square filter centered at frequency $\pm\omega_0$ and having a width $\Delta\omega \ll \omega_0$:

$$\begin{aligned} I_f(t) = & \frac{1}{\sqrt{2\pi}} \int_{-\omega_0-\Delta\omega/2}^{-\omega_0+\Delta\omega/2} e^{-i\omega t} I(\omega) d\omega \\ & + \frac{1}{\sqrt{2\pi}} \int_{\omega_0-\Delta\omega/2}^{\omega_0+\Delta\omega/2} e^{-i\omega t} I(\omega) d\omega \end{aligned} \tag{9}$$

This can be written as:

$$I_f(t) = I_1(t)\cos(\omega_0 t) + I_2(t)\sin(\omega_0 t) \tag{10}$$

where we defined the Hermitian *signal components* by:

$$I_1(t) = \frac{1}{\sqrt{2\pi}} \int_{-\Delta\omega/2}^{\Delta\omega/2} d\omega \, [I(\omega_0+\omega)e^{-i\omega t} + I^\dagger(\omega_0+\omega)e^{i\omega t}] \tag{11}$$

and

$$I_2(t) = \frac{-i}{\sqrt{2\pi}} \int_{-\Delta\omega/2}^{\Delta\omega/2} d\omega \, [I(\omega_0+\omega)e^{-i\omega t} - I^\dagger(\omega_0+\omega)e^{i\omega t}] \, . \tag{12}$$

For I_1 and I_2 to play the role of x and p above, one should obtain their commutator. In bosonic systems, I_1 and I_2 are linear in bosonic creation and annihilation operators and therefore their commutator is a c-number. For example, the transform of the current field propagating along a semi-infinite ideal transmission line having impedance R is given by[14][15]:

$$I(\omega) = -i\sqrt{\frac{2\hbar\omega}{R}} a(\omega) \tag{13}$$

where $a(\omega)$, is the annihilation operator of the transmission line mode at frequency ω. $a(\omega)$ satisfies the Bose commutation relations $[a(\omega), a^\dagger(\omega')] = \delta(\omega-\omega')$. Denoting $\Delta\nu = \frac{\Delta\omega}{2\pi}$ and using Eqs. (11)-(13) one can verify that the commutator of the components of this current is indeed a c-number and is given by

$$[I_1(t), I_2(t)] = i4\hbar\omega_0 \frac{1}{R} \Delta\nu. \tag{14}$$

Consider now an amplifier connected to a signal source at its input port and to a load resistor at its output port (Fig. 1.a). To avoid the specific details of the source, it is modelled by its quantum Norton equivalent circuit [16], i.e., by a resistor R_s in parallel to a current source $I_s(t)$ where R_s is the total resistance of the signal source between its two terminals and $I_s(t)$ is the *short-circuited current operator*, i.e., it is the Heisenberg current operator of the signal source if the amplifier is replaced by a short (Fig. 1.b). When coupled to an amplifier having an impedance R_{in} at its input port, the source $I_s(t)$ delivers a current $I_{in}(t)$ into it. If R_{in} is not a short than only a part of $I_s(t)$, namely $I_{in}(t) = I_s(t)R_s/(R_s+R_{in})$, will be delivered into the amplifier. Similarly, the combined system of the signal source and the amplifier acts as a resistance R_a in parallel to a current source $I_a(t)$ (Fig. 1.c) which delivers a current $I_{out}(t) = I_a(t)R_a/(R_a+R_\ell)$ to a load resistor R_ℓ. In analogy with Eq. (3) we write:

$$I_{out}(t) = G\left(\frac{R_s}{R_\ell}\right)^{1/2} I_{in}(t) + I_N(t) \tag{15}$$

where G^2 is the power gain and $I_N(t)$ is a noise operator which commutes with $I_{in}(t)$. The currents $I_{out}(t)$ and $I_{in}(t)$ are observable and, hence Hermitian. As a consequence $I_N(t)$ is also Hermitian. Written in terms of the signal components Eq. (15) becomes:

$$I_{out,q}(t) = G\left(\frac{R_s}{R_\ell}\right)^{1/2} I_{in,q}(t) + I_{Nq}(t) \quad q = 1, 2. \tag{16}$$

As in Eq. (4) the noise is taken to be uncorrelated with the signal. Consequently:

$$[I_{in,\beta}(t), I_{N\beta'}(t)] = 0 \qquad \beta, \beta' = 1, 2 \, . \tag{17}$$

Assuming I_s and I_a obey the bosonic relation, Eq. (14), with R replaced by R_s and R_a respectively, one has

$$[I_{\beta 1}(t), I_{\beta 2}(t)] = i4\hbar\omega_0 \frac{1}{R_\beta} \Delta\nu \qquad \beta = a, s \, . \tag{18}$$

We shall restrict ourselves to determining the optimum performance of power amplification when maximum

power is transferred from the signal source to the amplifier and from the amplifier to the load. In this case the amplifier should be impedance-matched to the source and the load and therefore one has $R_s = R_{in}$ and $R_a = R_\ell$. This implies that the current sources I_s and I_a deliver only half of their current to the amplifier input and the load, i.e., $I_s = 2I_{in}$ and $I_a = 2I_{out}$. Thus, Eq. (18) yields

$$[I_{in,1}(t), I_{in,2}(t)] = i\hbar\omega_0 \frac{1}{R_s}\Delta\nu$$
$$[I_{out,1}(t), I_{out,2}(t)] = i\hbar\omega_0 \frac{1}{R_\ell}\Delta\nu \ . \qquad (19)$$

Replacing $x,\ p,\ X$ and P by $I_{in,1},\ I_{in,2},\ I_{out,1}$ and $I_{out,2}$, using the commutator Eq. (19) instead of Eq. (2) and following the steps leading to Eq. (6) one obtains the well-known constraint on the minimal noise that the bosonic amplifier must add to the signal[6][7][11]:

$$\Delta I_{N1}(t)\Delta I_{N2}(t) \geq (G^2-1)\frac{\hbar\omega_0}{2}\frac{1}{R_\ell}\Delta\nu \ . \qquad (20)$$

In the derivation of Eq. (20) an essential use was made of the fact that the commutation relation of the bosonic currents, Eq. (19), was a c-number as in Eq. (2). However, as explained above, for other types of signals this is not the case. That is, one generally has

$$[I_1(t), I_2(t)] \neq c\ number \ , \qquad (21)$$

in which case the derivation given above is not justified.

We shall now derive generalized constraints on amplifier noise performance which will be valid regardless of whether the commutators of the components of the current operator are c-numbers or operators. In particular, these constraints are valid for both fermionic and bosonic currents. We assume that the amplifier and the signal are both in stationary states and again take the input-output relations, Eq. (16), except that the *differential conductances* (see below) of the amplifier and the source, g_s and g_ℓ, replace the ordinary ones, R_s^{-1} and R_ℓ^{-1} :

$$I_{out,q}(t) = G\left(\frac{g_\ell}{g_s}\right)^{1/2} I_{in,q}(t) + I_{N\beta}(t) \quad q=1,2 \ . \qquad (22)$$

Since the signals under consideration include those with the property Eq. (21), we do not have a simple commutator to work with. However, Kubo's fluctuation-dissipation theorem, generalized to nonequilibrium steady states, provides a means of carrying out the analysis by supplying a substitute for such a commutator. This is the key element in our derivation of the generalized constraints. Denoting

$$S(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t}\langle I(0)I(t)\rangle \ , \qquad (23)$$

the Kubo theorem states that [17]-[20]

$$S(-\omega) - S(\omega) = 2\hbar\omega g(\omega) \qquad \omega > 0 \qquad (24)$$

where $g(\omega)$ is the differential conductance of the system at frequency ω. Simple derivations of Eq. (24) are given in Refs. [21],[20]. An important aspect of Eq.(24) is that it holds regardless of whether the current is carried by bosons or fermions. Another important aspect of it is that it holds for *any* stationary state including nonequilibrium ones[22]. The current appearing in Eq. (23) may have a non-zero expectation value, $\langle I(t)\rangle \neq 0$, since the system is not necessarily in equilibrium. If one applies a small AC voltage, $V_{AC} = Ve^{i\omega t}$ (on top of any other field that may already exist and is driving the system out of equilibrium), then an additional current proportional to V_{AC} appears: $\langle I_V(t)\rangle = \tilde{\sigma}(\omega)Ve^{i\omega t}$. The differential conductance is defined as the real part of the proportionality coefficient, $\tilde{\sigma}(\omega)$, $g(\omega) \equiv Re\tilde{\sigma}(\omega)$.

We now derive some expressions that follow from Eqs. (23) and (24). First, we combine them into:

$$\int_{-\infty}^{\infty} d\tau e^{i\omega\tau}\langle [I(\tau), I(0)]\rangle = 2\hbar\omega g(\omega) \ . \qquad (25)$$

Recalling the assumption of stationarity, one can show that Eq. (23) implies:

$$\langle I(\omega)I(\omega')\rangle = \delta(\omega+\omega')S(-\omega) \qquad (26)$$
$$\langle I(\omega)I^\dagger(\omega')\rangle = \delta(\omega-\omega')S(-\omega) \qquad (27)$$
$$\langle I^\dagger(\omega)I(\omega')\rangle = \delta(\omega-\omega')S(\omega) \qquad (28)$$
$$\langle I^\dagger(\omega)I^\dagger(\omega')\rangle = \delta(\omega+\omega')S(\omega) \qquad (29)$$
$$\langle [I(\omega), I(\omega')]\rangle = \langle [I^\dagger(\omega), I^\dagger(\omega')]\rangle = 0, \quad \omega,\omega' > 0. \ (30)$$

From these relationships and Eq. (24) one also has

$$\langle [I(\omega), I^\dagger(\omega')]\rangle = 2\delta(\omega-\omega')\hbar\omega g(\omega) \ . \qquad (31)$$

From Eqs. (11), (12), (30), and (31) one obtains (compare with Eq. (14)):

$$\langle [I_1(t), I_2(t)]\rangle = 4i\hbar\omega_0 g\Delta\nu \ . \qquad (32)$$

where we assumed for simplicity that g is independent of the frequency. The generalization to the frequency-dependent case is straightforward.

To be able to carry out an amplifier noise analysis we first have to identify the relation between the currents appearing in Eqs. (22) and (32). As in Eqs. (23) and (24), the current and the conductance appearing in Eq. (32) are the Heisenberg current operators and the differential conductance of the system obtained when the voltage, or chemical potential is not allowed to fluctuate (as is the case when a 2-terminal device is connected across a zero impedance device such as a short or a voltage source). By their definition these are just, respectively, the currents produced by the current sources and the differential conductances of the resistors in the Norton circuits described above Eq. (15). Therefore, Eq. (32) applies for I_s and I_a:

$$\langle [I_{\beta 1}(t), I_{\beta 2}(t)]\rangle = 4i\hbar\omega_0 g_\beta \Delta\nu \qquad \beta = s, a \ . \qquad (33)$$

As above, assuming the impedance-matching conditions, $g_s = g_{in}$ and $g_a = g_\ell$, imply that the currents delivered to the amplifier and to the load are only half those of the Norton equivalent current generators, hence, $I_s(t) = 2I_{in}(t),\ I_a(t) = 2I_{out}(t)$. Eq. (33) now becomes:

$$\begin{aligned}\langle[I_{in,1}(t), I_{in,2}(t)]\rangle &= i\hbar\omega_0 g_s \Delta\nu \\ \langle[I_{out,1}(t), I_{out,2}(t)]\rangle &= i\hbar\omega_0 g_\ell \Delta\nu \ .\end{aligned} \tag{34}$$

Eqs. (17), (22) and (34) yield:

$$\langle[I_{N2}(t), I_{N1}(t)]\rangle = i(G^2-1)\hbar\omega_0 g_\ell \Delta\nu \ . \tag{35}$$

For the next step of the derivation we recall that for any two Hermitian operators A and B one has

$$\Delta A \Delta B \geq \frac{1}{2}|\langle[A, B]\rangle|, \tag{36}$$

where $[A, B]$ may be either an operator or a c-number. It follows that

$$\Delta I_{N2} \Delta I_{N1} \geq (G^2-1)\frac{\hbar\omega_0}{2} g_\ell \Delta\nu \ . \tag{37}$$

Eq. (37) is our main result. It is valid regardless of whether the signal is carried by fermions or bosons. In particular, it does not require the commutator of the current components to be a c-number, since only the expectation value of such a commutator, Eq. (32), enters into the derivation. For specific fermionic devices, such as mesoscopic or molecular amplifiers, Eqs. (35) and (37) allow one to obtain constraints on device parameters that must be met in order for the device to achieve quantum limited noise performance [24].

The expectation value of the time averaged noise power delivered to the load in the frequency band of width $\Delta\omega$ about ω_0 is given by

$$P_{\ell N} = \frac{1}{g_\ell}\langle I_N^2(t)\rangle = \frac{1}{2g_\ell}[\langle I_{N1}^2\rangle + \langle I_{N2}^2\rangle]. \tag{38}$$

Taking the noise to have zero mean and the the amplifier to be truly phase insensitive so that $\Delta I_{N2} = \Delta I_{N1}$, Eq. (37) yields

$$P_{\ell N} \geq \frac{\hbar\omega_0}{2}(G^2-1)\Delta\nu \ . \tag{39}$$

Using Eq. (33) it is also straightforward to demonstrate that it is permissible for an amplifier to noiselessly carry out the transformation:

$$\begin{aligned}I_{out,1}(t) &= G\sqrt{\frac{g_\ell}{g_s}} I_{in,1}(t) \\ I_{out,2}(t) &= \frac{1}{G}\sqrt{\frac{g_\ell}{g_s}} I_{in,2}(t).\end{aligned} \tag{40}$$

In conclusion, using the nonequilibrium Kubo formula, one can generalize the usual noise limits of amplifiers to the case in which the commutator of the signal components is not a c-number as, e.g., is the case of a current carried by fermions. Again, for phase insensitive amplifiers, a noise power of $(G^2-1)\hbar\omega/2$ per unit bandwidth must be added to the amplified signal. Furthermore, this analysis allows for the possibility of phase sensitive fermion based devices which contributes no noise to the component of the signal that is amplified.

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